Physics 141: Exam II

University of Rochester
Department of Physics & Astronomy

Instructions

There are two problems! You must do both.
Questions in each problem are marked with falling stars. 1☄ < 5min of work.
The test takes 80min, so there are 16 ☄s in total, 8 in each of the two problems.
Later questions depend on answers from earlier questions, so alot your time properly.
Show your thinking - partial credit may be earned everywhere.
Write legibly with a font size greater than 10.
Donuts are on their way!
Excelsior!

Name ____

P1	☄	
	☄	
	☄	
	☄☄	
	☄☄	
	☄	
P2	☄☄	
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	☄	

$G = 6.674\times10^{-11}\dfrac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$ $c = 2.998\times10^{8}\dfrac{\text{m}}{\text{s}}$ $h = 6.626\times10^{-34}\text{J}\cdot\text{s}$ $k_B = 1.381\times10^{-23}\dfrac{\text{kg}\cdot\text{m}^2}{\text{s}^2\cdot\text{K}}$

$M_\oplus = 5.972\times10^{24}\text{kg}$ $M_\odot = 1.989\times10^{30}\text{kg}$ $R_\oplus = 6.371\times10^{6}\text{m}$ $R_\odot = 6.957\times10^{8}\text{m}$

$AU = 1.496\times10^{11}\text{m}$

	Cartesian Coordinates	Polar Coordinates	Cartesian	⇔ Polar
	$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$	$\boldsymbol{r} = r\hat{\boldsymbol{r}}$	$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}}$	$\dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$	$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$	$y = r\sin\theta$	$\theta = \arctan\frac{y}{x}$
$\boldsymbol{a} = \dfrac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}}$	$\ddot{\boldsymbol{r}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}$	$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$		

Newton's 2nd Law :	$\boldsymbol{F} = \dfrac{d\boldsymbol{p}}{dt}$	Momentum :	$\boldsymbol{p} = m\boldsymbol{v}$
Impulse :	$\Delta\boldsymbol{p} = \int \boldsymbol{F}dt$	Universal Gravitation :	$\boldsymbol{F}_g = -G\dfrac{m_1m_2}{\lvert\boldsymbol{r}_2-\boldsymbol{r}_1\rvert^3}(\boldsymbol{r}_2-\boldsymbol{r}_1)$
Work :	$W = \int_C \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}$	Kinetic Energy:	$K = \dfrac{1}{2}mv^2$
Work-Energy Theorem:	$W = \Delta K$	Conservative Force :	$\boldsymbol{F} = -\nabla U$
Spring Potential Energy :	$U_s = \dfrac{1}{2}m\omega^2\lvert\boldsymbol{r}-\boldsymbol{r}_{eq}\rvert^2$	Potential Energy :	$U(\boldsymbol{r}_f) - U(\boldsymbol{r}_i) = -\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$
Grav. Potential Energy :	$U_g = -G\dfrac{m_1m_2}{\lvert\boldsymbol{r}_2-\boldsymbol{r}_1\rvert}$	Mass of object :	$M = \int_\Omega \rho(\boldsymbol{r})dV$
Center of mass:	$\boldsymbol{R} = \dfrac{1}{M}\sum_i m_i\boldsymbol{r}_i$	or $\dfrac{1}{M}\int_\Omega \rho(\boldsymbol{r})\boldsymbol{r}dV$	
Angular Momentum:	$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} = \tilde{\boldsymbol{I}}\cdot\boldsymbol{\omega}$	Torque:	$\boldsymbol{\tau} = \boldsymbol{r}\times\boldsymbol{F} = \tilde{\boldsymbol{I}}\cdot\boldsymbol{\alpha}$

Vector Identities

$\boldsymbol{A}\cdot\boldsymbol{B} = \lvert\boldsymbol{A}\rvert\lvert\boldsymbol{B}\rvert\cos\theta$ $\lvert\boldsymbol{A}\times\boldsymbol{B}\rvert = \lvert\boldsymbol{A}\rvert\lvert\boldsymbol{B}\rvert\sin\theta$ $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$

Expanding δx around x $f(x+\delta x) = f(x) + f'(x)\delta x + \dfrac{1}{2!}f''(x)\delta x^2 + \cdots$

Taylor Expansions

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$ $\ln\lvert 1+x\rvert = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$ $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \cdots$ $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2}x^2 + \cdots$

Problem 1 A one-dimensional rod is located on the x-axis so that its endpoints are at $x = 0$ and $x = L$. The rod has a linear density

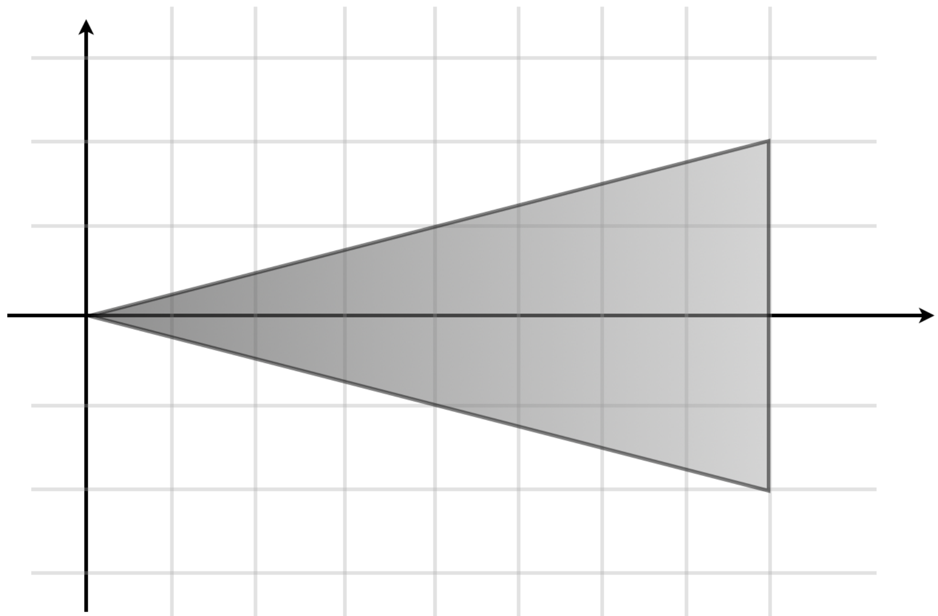
$$\lambda(x) = 12\frac{m}{L}\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

for $0 \le x \le L$. m is a positive constant. ( ) Sketch the density profile of the rod. Label relevant points on both axes. ( ) What is the mass of a tiny piece of the rod located between x and $x + dx$? Add up the masses of all the pieces to find the total mass of the rod. The answer should be very simple if you did everything right. ( ) Where is the rod's center of mass located? Show your calculation, and use your sketch to check your answer.

A two-dimensional sheet of matter in the shape of a triangle has vertices at $(0, 0)$, $(2L, L/2)$, and $(2L, -L/2)$. The surface density of the sheet is

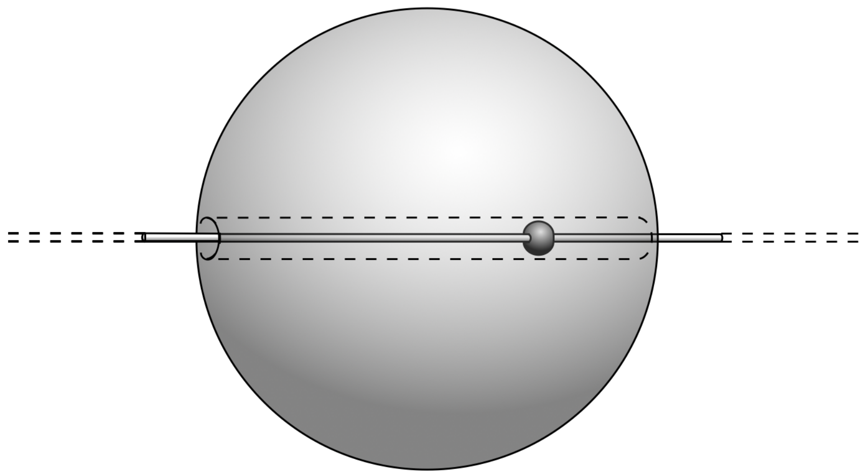
$$\sigma(x, y) = 8\frac{m}{L}\frac{1}{x} \tag{1}$$

inside the triangle. m is a positive constant. () Find the total mass of the sheet. Set up the integral with proper bounds and evaluate it. Use the diagram to help you think about the problem. Labeling directly on the diagram is useful. If you do everything right your answer should be very simple. () Find the location of the sheet's center of mass. Again, your answer should come out very simple.



The m's in the rod and triangle problem are equal. Someone standing a horizontal distance $d = 35$m away from you throws the triangle from the last problem vertically into the air. You, an expert archer, grab the rod from the first problem and use it as an arrow, firing it at the triangle. At the **peak** of both the arrow and triangle's trajectories, the two collide, the arrow piercing the triangle so that their centers of mass coincide. (💣) Neglecting air resistance, how far away from you will the resulting object land on the ground? If you're doing any kinematics you're over-thinking it. Drawing a picture always helps.

Problem 2 A bead of mass m is confined to move on an infinite horizontal rod. A large sphere of constant density, radius R, and mass $M \gg m$ is built around a part of the rod. A hole is left in the sphere large enough for the bead to slide freely through the sphere. It passes through the center of the sphere, but is not large enough to have any significant impact on M. Use the variable r to describe the distance the bead is along the rod from the center of the sphere.



(💣💣) Using the shell theorem determine the gravitational force felt by the bead everywhere along the rod. The constant density of the sphere is a clue for how to do this inside the sphere. Your answer will have different algebraic forms when $r < R$ and when $r > R$. Write it using the sphere's surface gravity, $g = GM/R^2$, so that the solution is very simple.

(🌶🌶🌶) Integrate your result to find the gravitational potential energy along the entire rod. For the constant of integration you get integrating the $r > R$, set the potential energy infinitely far away to 0. For the constant from $r < R$, make sure the total solution is continuous at $r = R$.

(🌶) You place the bead at a the point $r = 2R$ and let go of it at rest. The bead moves down the potential gaining kinetic energy. As it passes the center of the sphere, what speed does it have? There are no frictional forces present.

You placed a second bead of mass m at rest at the center of the sphere. When the bead from the last part zips past the center, it collides with the second bead, sticking to it. () What is the speed of the combined super-bead the moment after the collision? () How far does the super-bead move along the rod before coming to a halt?