Newton's Shell Theorem

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[1]*Earth 42*

Prop. LXXI. Theor. XXXI.

Iiſdem poſitis, dico quod corpuſculum extra Sphæricam ſuperficiem conſtitutum attrahitur ad centrum Sphæræ, vi reciproce proportionali quadrato diſtantiæ ſuæ ab eodem centro.

In 1687 Isaac Newton published his groundbreaking *Philosophiæ naturalis principia mathematica*, simply referred to as *the Principia*. The magnum opus spans three books in which he lays out eight definitions, his three axioms (laws of motion), and follows with some 200 propositions proving various theorems. In book I, his 71$^{\text{st}}$ proposition (31$^{\text{st}}$ theorem) is

> *The same things supposed as above, I say, that a corpuscle placed without the spherical surface is attracted towards the center of that sphere with a force reciprocally proportional to the square of its distance from that center.*

now famously referred to, together with the 32$^{\text{nd}}$ theorem, as the *shell theorem*. The theorem concerns the strength of the gravitational field produced by a thin spherical shell, hence the name *shell* theorem. It tells us that outside the shell the field is indistinguishable from that of a point mass at the same location as the center of the sphere, and inside the shell the field vanishes!

Here we prove the theorem using modern notation, and hope that the reader finds it both readable and illuminating. The proof of is instructive, showing how to ingeniously slice up the shell and calculate the contribution of each little piece to the total field. Taking advantage of symmetry, one can then add up all the tiny contributions to find the exact value of the field. Let's begin!

THEOREM

Consider a spherical shell of mass M and radius R, and a point $\mathcal{P}$ a distance r away from the center of the shell. The gravitational field produced by the shell at point $\mathcal{P}$ is

$$\boldsymbol{g} = \begin{cases} -\dfrac{GM}{r^2}\hat{\boldsymbol{r}} & r > R \\ \boldsymbol{0} & 0 \le r < R \end{cases}. \tag{1}$$

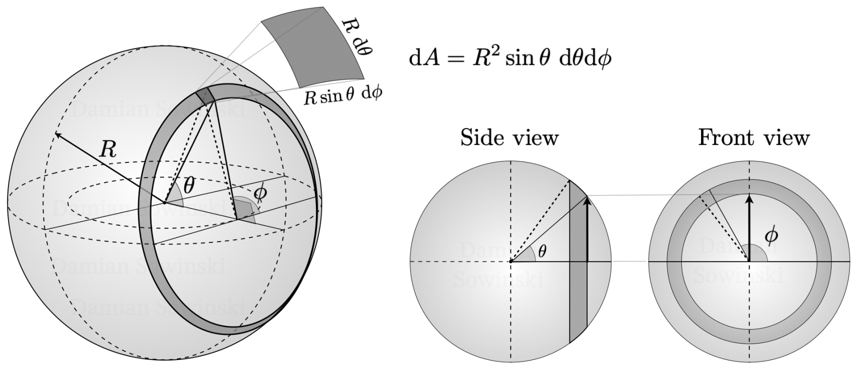
PROOF

The shell is idealized to be a truly two-dimensional object, but imaging a tiny thickness doesn't alter the results that follow. In this idealization, the shell is uniform, so that its surface mass density, σ, is constant. The value is the mass of the shell divided by the total surface area, namely

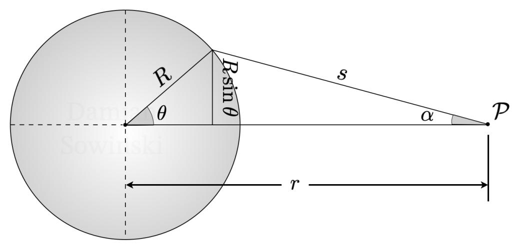
$$\sigma = \frac{M}{4\pi R^2}. \tag{2}$$

This is useful because it allows one to assign to each small area element of the surface, $\mathrm{d}A$, a small mass, $\mathrm{d}m = \sigma \mathrm{d}A$.

How should we slice up the shell into pieces to make calculating the field as easy as possible? The first step is to divide the shell up into rings as shown below. These rings are then cut up into tiny area elements. Study the figure until you are convinced the little surface elements have the area that they do.



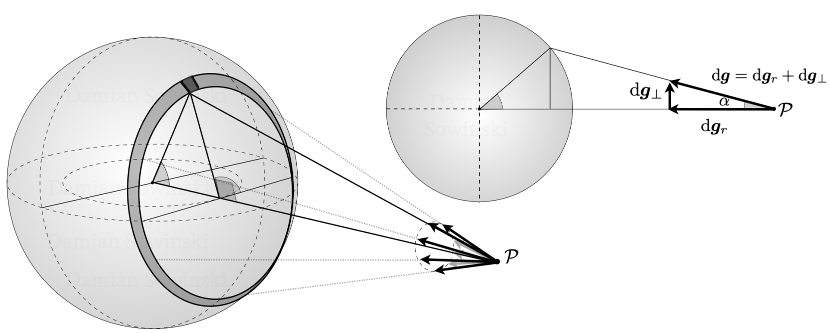
First we calculate the gravitational field outside the shell. By spherical symmetry all points with the same $r > R$ experience a gravitational field of the same magnitude, so choose a point $\mathcal{P}$ lined up on the horizontal axes, as in the next figure. Now consider the ring at the angle θ. Every little mass element on that ring is the same distance from



our chosen point, namely s. Thus the mass element on the ring will generate a gravitational field of magnitude

$$\mathrm{d}g = G\frac{\mathrm{d}m}{s^2} = \frac{GM}{4\pi}\frac{\sin\theta \mathrm{d}\theta\mathrm{d}\phi}{s^2} \tag{3}$$

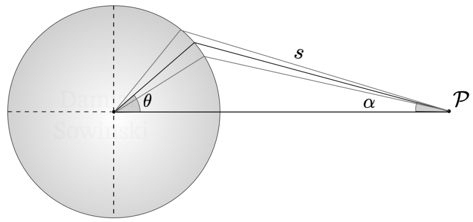
To do the ϕ integral note that only the direction of the gravitational field depends on it. For every value of ϕ one can break up the vector representing the gravitational field as $\mathrm{d}\boldsymbol{g} = \mathrm{d}\boldsymbol{g}_r + \mathrm{d}\boldsymbol{g}_\perp$ as in the next figure. Notice that



mass element on the other side of the ring will produce the same $\mathrm{d}\boldsymbol{g}_r$ but the perpendicular component will be flipped, $-\mathrm{d}\boldsymbol{g}_\perp$. Adding up (doing the ϕ integral) all the gravitational field contributions coming from the ring we see that the perpendicular components will cancel, leaving only the radial component. Therefore the gravitational field generated by the entire ring will point radially inward and have magnitude

$$\mathrm{d}g_r = \frac{GM}{4\pi}\frac{\sin\theta\mathrm{d}\theta}{s^2}\int_0^{2\pi}\mathrm{d}\phi\cos\alpha = \frac{GM}{2}\frac{\cos\alpha\sin\theta}{s^2}\mathrm{d}\theta \tag{4}$$

We're not done yet. We now have to cut up the rest of the shell into similar rings and add up the contributions coming from each one. This amounts to doing the θ integral. Unfortunately equation 4 is not expressed solely in terms of θ. Both α and s implicitly depend on θ. Changing any one of the three must be compensated for by a change in the other two, which is seen in the next figure. It turns out that choosing to write everything in terms of s



makes integrating equation 4 tractable. To do that we will need to write $\sin\theta$, $\mathrm{d}\theta$, and $\cos\alpha$ in terms of s.

To accomplish the above feat we must recall the law of cosines for both the angles:

$$R^2 = s^2 + r^2 - 2sr\cos\alpha \tag{5}$$
$$s^2 = R^2 + r^2 - 2Rr\cos\theta. \tag{6}$$

The first of these, equation 5, can easily be solved for $\cos\alpha$,

$$\cos\alpha = \frac{s^2 + r^2 - R^2}{2sr}. \tag{7}$$

Meanwhile, the second, equation 6, must be differentiated. Remember that only s and θ are variables, while R and r are fixed. One finds

$$2s\mathrm{d}s = 2Rr\sin\theta\mathrm{d}\theta \tag{8}$$

which kills two birds with one stone. Rearranging

$$\sin\theta\mathrm{d}\theta = \frac{s}{Rr}\mathrm{d}s \tag{9}$$

We can now plug in results 7 and 9 into equation 4

$$\begin{aligned}\mathrm{d}g_r &= \frac{GM}{2}\frac{1}{s^2}\frac{s^2 + r^2 - R^2}{2sr}\frac{s}{Rr}\mathrm{d}s\\ &= \frac{GM}{4Rr^2}\left(1 + \frac{r^2 - R^2}{s^2}\right)\mathrm{d}s\end{aligned} \tag{10}$$

We're almost there; all we have to do now is add up the contribution from each ring. What are the limits of integration? The closest ring has $s = r - R$, while the farthest has $s = r + R$. Thus

$$\begin{aligned}g(r) &= \frac{GM}{4Rr^2}\int_{r-R}^{r+R}\left(1 + \frac{r^2 - R^2}{s^2}\right)\mathrm{d}s\\ &= \frac{GM}{4Rr^2}\left[s - \frac{r^2 - R^2}{s}\right]_{r-R}^{r+R}\\ &= \frac{GM}{4Rr^2}\left(r + R - \frac{r^2 - R^2}{r + R} - \left(r - R - \frac{r^2 - R^2}{r - R}\right)\right)\\ &= \frac{GM}{4Rr^2}(4R)\\ &= \frac{GM}{r^2}\end{aligned} \tag{11}$$

But that's exactly the strength of the gravitational field produced by a point mass a distance r away from $\mathcal{P}$! Apparently the extended nature of the shell does not matter.

What about the field on the inside of the shell? The field created by each ring doesn't change, so to add up all their contributions we need to integrate the same integrand again. All that has changed are the limits of integration.

Now the shortest value of s is $R-r$, and its longest value is $R+r$. Plugging the new limits gives

$$\begin{aligned} g(r) &= \frac{GM}{4Rr^2}\int_{R-r}^{R+r}\left(1+\frac{r^2-R^2}{s^2}\right)\mathrm{d}s \\ &= \frac{GM}{4Rr^2}\left(R+r-\frac{r^2-R^2}{R+r}-\left(R-r-\frac{r^2-R^2}{R-r}\right)\right) \\ &= \frac{GM}{4Rr^2}\left(R+r+R-r-(R-r+R+r)\right) \\ &= 0. \end{aligned} \tag{12}$$

The field vanishes, as promised. Q.E.D.