Eigensystems of Circulant Matrices

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INTRODUCTION

In these notes we endeavor to understand the spectrum of a special class of linear operators known as circulants. In particular, is there anything special about the spectrum of these operators? Their singular value decomposition? What is their relationship to other types of transforms? Before we begin, a few mathematical preliminaries are covered in the appendices. These include a review of roots of unity and their relationship to discrete Fourier transforms. If the reader is comfortable with those topics, it's still good to check out the normalization conventions used herein.

Definition

Consider a finite sequence of elements from some field $\mathbb{F}$, namely $c_0, c_1, \ldots, c_{N-1}$. The **circulant** of this sequence, $\boldsymbol{C} \in \mathbb{F}^{N\times N}$, is constructed by setting the components

$$[\boldsymbol{C}]_{nm} = c_{(m-n) \bmod N}. \tag{1}$$

Explicitly, the circulant is written by shifting and stacking the sequence to form a square matrix,

$$\boldsymbol{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ c_{N-2} & c_{N-1} & c_0 & \cdots & c_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix} \tag{2}$$

Note that some authors define the circulant as the transpose of the above, or by permuting the sequence to read the other way. The following results are insensitive to these particulars.

THE CIRCULANT SPECTRUM

What are the eigenvalues and eigenvectors of a circulant? The surprisingly neat answer to this question is that every circulant has the exact same eigenvectors! Different circulants differ only in their spectrum of eigenvalues.

Theorem Circulant Spectrum

For a circulant $\boldsymbol{C} \in \mathbb{C}^{N\times N}$ constructed from the components of $\boldsymbol{c} \in \mathbb{C}^N$,

1. the eigenvectors, $\boldsymbol{e}_n$, do not depend on $\boldsymbol{c}$, and have components $[\boldsymbol{e}_n]_m = N^{-1/2}\omega_N^{-nm}$. The eigenvector matrix $\boldsymbol{E} = [\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{N-1}]$ is unitary
2. the eigenvalue corresponding to the eigenvector is $\boldsymbol{e}_n$ is $\lambda_n = [\tilde{\boldsymbol{c}}]_n$, where $\tilde{\boldsymbol{c}} = \mathcal{F}[\boldsymbol{c}]$ is the DFT of $\boldsymbol{c}$ and the $A = \sqrt{N}$ normalization convention has been used for the DFT.

If you're unfamiliar with discrete Fourier transforms, go look a the appendix.

Proof. It is easy to see that every circulant will share at least one eigenvector, namely $\boldsymbol{e}_0 = [1, 1, \ldots, 1]^T$. The corresponding eigenvalue is $\lambda_0 = c_0 + c_1 + \cdots + c_{N-1}$. This one eigenvector gives us a clue to what the other ones may be. Examining the DFT operator, $\boldsymbol{\mathcal{F}}$, in eq.(8), notice that $\boldsymbol{e}_0$ is proportional to its first column. Perhaps we should see what the circulant does to the other columns?

To that end, consider the n^{th} column of $\boldsymbol{E}$, the vector with components $[\boldsymbol{e}_n]_m = [\boldsymbol{E}]_{mn} = N^{-1/2}\omega_N^{-nm}$. What does the circulant do to this vector?

$$\begin{aligned}
[\boldsymbol{C}\cdot\boldsymbol{e}_n]_m &= \sum_{k=0}^{N-1} [\boldsymbol{C}]_{mk}[\boldsymbol{e}_n]_k \\
&= \sum_{k=0}^{N-1} c_{(k-m)\bmod N}\ \frac{1}{\sqrt{N}}\omega_N^{-kn} \\
&= \sum_{k=0}^{N-1} c_{(k-m)\bmod N}\ \frac{1}{\sqrt{N}}\omega_N^{-(k-m)n}\omega_N^{-mn} \\
&= \left(\sum_{k=0}^{N-1} \omega_N^{-nk} c_k\right)\frac{1}{\sqrt{N}}\omega_N^{-mn} \\
&= \left(\frac{\sqrt{N}}{A}[\tilde{\boldsymbol{c}}]_n\right)[\boldsymbol{e}_n]_m
\end{aligned}$$

where we have used $c_{(-j)\mathrm{mod}N} = c_{N-j}$ and $\omega_N^{kn} = \omega_N^{-(N-k)n}$ in going from the third to fourth line.

We see that $\boldsymbol{e}_n$ is an eigenvector of the circulant, and the corresponding eigenvalue is (proportional to) the n^{th} component of the DFT of the sequence from which the circulant was constructed. The structure of the eigenvectors is independent of the sequence, but the eigenvalues are not, confirming the claim. Also see the reason for keeping the arbitrary constant in the DFT - choosing $A = \sqrt{N}$ fixes the scale of the eigenvalues.

One can also check the orthonormality relation between the eigenvectors,

$$\begin{aligned}
\boldsymbol{e}_n^\dagger \cdot \boldsymbol{e}_m &= \sum_{k=0}^{N-1} [\boldsymbol{e}_n^\dagger]_k[\boldsymbol{e}_m]_k \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \omega_N^{nk}\omega_N^{-km} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(n-m)/N} \\
&= \delta_{nm} \qquad\qquad \text{(Orthonormality Relation)}
\end{aligned}$$

where in the last line we observe that if $n \neq m$, the sum of all the roots of unity vanishes (see appendix); if $n = m$ then the sum is N.

Orthonormality imlies that these vectors are independent. It is also a good exercise to see that these vectors are also *complete*. *Completeness* is a fancy way of saying that, taken together, they resolve the identity matrix,

$$\begin{aligned}
\left[\sum_{k=0}^{N-1} \boldsymbol{e}_k\boldsymbol{e}_k^\dagger\right]_{nm} &= \sum_{k=0}^{N-1} [\boldsymbol{e}_k]_n[\boldsymbol{e}_k^\dagger]_m \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \omega_N^{-nk}\omega_N^{mk} \\
&= \delta_{nm} \\
&= [\mathbb{1}]_{nm}
\end{aligned}$$

Together, independence and completeness make the set of eigenvectors a basis (an orthonormal one at that!).

Completeness shows us that $\boldsymbol{E}$ is unitary,

$$\begin{aligned}[\boldsymbol{E}\cdot\boldsymbol{E}^\dagger]_{nm} &= \sum_{k=0}^{N-1}[\boldsymbol{E}]_{nk}[\boldsymbol{E}^\dagger]_{km} \\ &= \sum_{k=0}^{N-1}[\boldsymbol{e}_k]_n[\boldsymbol{e}_k]^*_m \\ &= [\mathbb{1}]_{nm}. \end{aligned} \tag{3}$$

completing, pun intended, the proof. □

APPENDICES

Roots of Unity

Definition

For any $N \in \mathbb{N}$ and $k \in \{0, 1, \ldots, N-1\}$, the N^{th} **roots of unity** are the set of complex numbers $\omega_N^k = e^{2\pi i k/N}$. We call the $k = 1$ case, ω_N, the **primitive** N^{th} **root**.

It is easy to verify that $(\omega_N^k)^N = 1$, and hence their name. The roots are arranged in the complex plane on the circle of radius 1, separated by angles of $2\pi/N$. It is a quick exercise to show that the sum of all the roots of unity, for fixed N, vanishes:

$$\sum_{k=0}^{N-1} \omega_N^k = \frac{1-\omega_N^N}{1-\omega_N} = 0 \tag{5}$$

In fact it isn't only the primitive root's geometric sum that vanishes. The geometric sum made out of any of the $k \neq 0$ roots vanishes, $\sum_n \omega_N^{kn} = 0$. The roots of unity show up in quite a few places, so it is prudent to gain some familiarity with their properties.

Discrete Fourier Transforms

Let's explore the connection between discrete Fourier transforms (DFT) and roots of unity. The DFT is a way to transform complex valued vectors into other complex valued vectors. The transformation picks out the "waviness" in the vector.

Definition

The **discrete Fourier transform (DFT)** is an invertible linear map,

$$\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N : \boldsymbol{x} \mapsto \tilde{\boldsymbol{x}} = \mathcal{F}[\boldsymbol{x}].$$

In component form it, and its inverse (**iDFT**), are

$$[\tilde{\boldsymbol{x}}]_n = \frac{A}{\sqrt{N}} \sum_{m=0}^{N-1} e^{-2\pi i \frac{nm}{N}} [\boldsymbol{x}]_m$$

$$[\boldsymbol{x}]_n = \frac{A^{-1}}{\sqrt{N}} \sum_{m=0}^{N-1} e^{2\pi i \frac{nm}{N}} [\tilde{\boldsymbol{x}}]_m$$

where A is an arbitrary constant.

Note that A is chosen based on preference. The default in MATLAB, and on most computational platforms, is $A = \sqrt{N}$. For analytical purposes, restoring the symmetry between the DFT and iDFT is accomplished by $A = 1$. In what follows we will keep it arbitrary.

Consider the operator $\boldsymbol{\mathcal{F}} \in \mathbb{C}^{N \times N}$ associated with the DFT. Its components are written using the roots of unity,

$$[\boldsymbol{\mathcal{F}}]_{nm} = \frac{A}{\sqrt{N}} \omega_N^{-nm} \tag{6}$$

$$[\boldsymbol{\mathcal{F}}^{-1}]_{nm} = \frac{A^{-1}}{\sqrt{N}} \omega_N^{nm} \tag{7}$$

Explicitly,

$$\boldsymbol{\mathcal{F}} = \frac{A}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \cdots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \cdots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \cdots & \omega_N^{-(N-1)(N-1)} \end{bmatrix} \tag{8}$$

Let's figure out some of the properties of $\boldsymbol{\mathcal{F}}$.

It's easy to see that $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}^T$, the operator is symmetric. How does it transform under conjugation?

$$\begin{aligned} [\boldsymbol{\mathcal{F}}^*]_{nm} &= \left(\frac{A}{\sqrt{N}} \omega_N^{-nm} \right)^* \\ &= \frac{A^*}{\sqrt{N}} \omega_N^{nm} \\ &= |A|^2 [\boldsymbol{\mathcal{F}}^{-1}]_{nm} \end{aligned}$$

Combined with symmetry means the transformation is almost unitary, $\boldsymbol{\mathcal{F}}^\dagger = |A|^2 \boldsymbol{\mathcal{F}}^{-1}$. This is why the $A = 1$ convention is preferred in quantum mechanics.