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In 1916 Einstein linearized his own field equations and found that a small disturbance in the metric propagates at the speed of light, obeying a wave equation sourced by the stress-energy of matter [1]. That weak-field limit is the natural arena for any mass whose gravity is gentle but whose motion need not be, and it is the arena of these notes. We compute the metric perturbation produced by a compact mass as it moves through Minkowski space — first drifting uniformly, then accelerating — and watch the gravitational field acquire, in turn, a gravito-magnetic part and a radiative one.

The notes fall into two parts. The first treats a mass in uniform motion, where the source is a rigid translation and the perturbation collapses to a single scalar potential times a velocity matrix. The second lifts that restriction: the mass accelerates, the retarded-time condition turns implicit, and the field separates — exactly as in electromagnetism — into a velocity near-field and an acceleration radiation-field, the gravitational Liénard–Wiechert construction [2, 3]. We close by specializing to constant proper acceleration, where the worldline is the Rindler hyperbola and the field ahead of the mass is compressed without bound against an ever-receding horizon [8, 9].

Throughout we keep SI units and carry every factor of  $G$  and  $c$  explicitly. The metric signature is  $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$ ,  $\rho$  denotes the mass density, and  $\bar{h}_{\mu\nu}$  is the trace-reversed perturbation defined below.

## The Uniformly Moving Mass

Consider a spherically symmetric mass distribution moving through Minkowski space. We wish to find the linear perturbation  $h_{\mu\nu}$  to the metric induced by this mass. First off we have the weak field Einstein equation, which in the Lorenz gauge reads [2, 4]

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}, \quad (1)$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}, \quad (2)$$

where  $\bar{h}_{\mu\nu}$  is the trace-reversed perturbation and, in the  $(+, -, -, -)$  signature,  $\partial_\alpha \partial^\alpha = c^{-2} \partial_t^2 - \nabla^2$ . We can invert (1) with the retarded Green's function of the d'Alembertian [4]. The result reads

$$\bar{h}_{\mu\nu}(\vec{x}, t) = \frac{4G}{c^4} \int d^3x' \frac{T_{\mu\nu}(\vec{x}', t_r)}{|\vec{x} - \vec{x}'|}, \quad (3)$$

$$t_r = t - \frac{|\vec{x} - \vec{x}'|}{c}. \quad (4)$$

For a mass moving uniformly along the  $x$ -direction with velocity  $v$ , the dust stress tensor is  $T^{\mu\nu} = \rho u^\mu u^\nu$  with  $u^\mu = \gamma(c, v, 0, 0)$ . Keeping the factors of  $c$  explicit,

$$T^{\mu\nu} = \rho \gamma^2 \begin{pmatrix} c^2 & cv & 0 & 0 \\ cv & v^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5)$$

The Lorentz factor  $\gamma = (1 - v^2/c^2)^{-1/2}$  is carried along explicitly here; the original note's matrix is the  $v \ll c$  limit, in which  $\gamma \rightarrow 1$  and the  $c$ 's in each entry track the index structure ( $T^{00} \sim \rho c^2$ ,  $T^{0i} \sim \rho c v$ ,  $T^{ij} \sim \rho v^2$ ). For uniform motion the source at retarded time is a rigid translation,  $\rho(\vec{x}', t_r) = \rho(\vec{x}' - \vec{v}t_r)$ , so the whole perturbation factors as a single scalar potential times the velocity matrix,

$$\bar{h}^{\mu\nu} = \frac{I(\vec{x} - \vec{v}t)}{c^2} \gamma^2 \begin{pmatrix} c^2 & cv & 0 & 0 \\ cv & v^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

$$I(\vec{x} - \vec{v}t) = \frac{4G}{c^2} \int d^3x' \frac{\rho(\vec{x}', t_r)}{|\vec{x} - \vec{x}'|}. \quad (7)$$

For a classical source only  $\bar{h}_{00}$  and  $\bar{h}_{0i}$  feed back into the metric at leading order — the gravito-electric potential and the gravito-magnetic potential, respectively.

## The Uniformly Accelerating Mass

We now ask what changes when the mass no longer drifts but accelerates. The machinery of (3)–(4) is untouched; what breaks is the rigid translation that let us pull the source out as a function of  $\vec{x} - \vec{v}t$ . Let the centre of the distribution follow a worldline  $\vec{x}_s(t)$  with velocity  $\vec{v}(t) = \dot{\vec{x}}_s(t)$  and acceleration  $\vec{a}(t) = \ddot{\vec{x}}_s(t)$ . The retarded condition becomes *implicit*,

$$t_r = t - \frac{|\vec{x} - \vec{x}_s(t_r)|}{c}, \quad (8)$$

since the field point now “sees” the mass where it was when the signal left, not at a position linear in  $t_r$ .

### Point source: the gravitational Liénard–Wiechert potentials

Take the small-body limit,  $\rho(\vec{x}', t) \rightarrow M \delta^3(\vec{x}' - \vec{x}_s(t))$ . The  $\delta$ -function collapses the integral in (3), but the retardation in (8) leaves a Jacobian. Define the separation  $\vec{R} = \vec{x} - \vec{x}_s(t_r)$ , the unit vector  $\hat{n} = \vec{R}/R$ , and the dimensionless retardation factor

$$\kappa = 1 - \frac{\hat{n} \cdot \vec{v}(t_r)}{c}. \quad (9)$$

Carrying the  $\delta$ -function through  $d^3x'$  contributes a factor  $1/\kappa$  — the same Doppler-like compression that appears in the electromagnetic Liénard–Wiechert construction [5]. The trace-reversed potential is then

$$\bar{h}^{\mu\nu}(\vec{x}, t) = \frac{4GM}{c^4} \frac{u^\mu u^\nu}{(-u^\alpha R_\alpha)} = \frac{4GM}{c^2} \frac{1}{\kappa R} \frac{u^\mu u^\nu}{c^2} \Big|_{t_r}, \quad (10)$$

with everything on the right evaluated at the retarded time  $t_r$ . Reading off the components that source the metric,

$$\bar{h}_{00} = \frac{4GM}{c^2} \frac{\gamma}{\kappa R} \Big|_{t_r}, \quad (11)$$

$$\bar{h}_{0i} = -\frac{4GM}{c^3} \frac{\gamma v_i}{\kappa R} \Big|_{t_r}. \quad (12)$$

The  $1/\kappa$  enhancement is the gravito-magnetic analog of the headlight effect: a mass accelerating toward the observer crowds its field lines, while one receding rarefies them.

### Velocity field and acceleration field

Differentiating the potentials to form the gravito-electric field  $\vec{g}$  splits it, exactly as in electromagnetism, into two pieces [6],

$$\vec{g} = \underbrace{-\frac{GM}{\gamma^2 \kappa^3 R^2} \left( \hat{n} - \frac{\vec{v}}{c} \right)}_{\text{velocity field } \sim R^{-2}} - \underbrace{\frac{GM}{c^2 \kappa^3 R} \hat{n} \times \left[ \left( \hat{n} - \frac{\vec{v}}{c} \right) \times \frac{\vec{a}}{c} \right]}_{\text{acceleration field } \sim R^{-1}} \Big|_{t_r}. \quad (13)$$

The velocity field is the boosted near-field, falling as  $R^{-2}$  and carrying no energy to infinity. The acceleration field falls only as  $R^{-1}$  and is proportional to  $\vec{a}$ ; it is the linearized signature of gravitational radiation. Note, however, that a single accelerating monopole cannot radiate — momentum conservation forbids it, just as charge conservation kills the dipole’s electromagnetic counterpart’s monopole — so on integrating over a physical source this  $R^{-1}$  term cancels against the recoil of whatever drives the acceleration. The expression above is the field of an *idealized* forced mass, useful as a building block, not a standalone radiator.

### Constant proper acceleration: the Rindler worldline

Specialize now to *constant proper acceleration*  $\alpha$  along  $x$ . The worldline is the hyperbola

$$x_s(t) = \frac{c^2}{\alpha} \sqrt{1 + \left( \frac{\alpha t}{c} \right)^2}, \quad v(t) = \frac{\alpha t}{\sqrt{1 + (\alpha t/c)^2}}, \quad a(t) = \frac{\alpha}{[1 + (\alpha t/c)^2]^{3/2}}. \quad (14)$$

This is the Rindler trajectory [8, 9] — the mass asymptotes to the light cone but never reaches  $c$ , and the coordinate acceleration  $a(t)$  bleeds away as  $t \rightarrow \pm\infty$  even though the proper acceleration  $\alpha$  is fixed. Substituting (14) into (11)–(12) gives the instantaneous potentials, and into (13) the field; the only subtlety is solving the implicit retarded condition (8) for  $t_r$ , which for the hyperbola is a quadratic and so closes in elementary form. For an observer on the  $x$ -axis ahead of the mass,  $\hat{n} = \hat{x}$  and

$$\kappa = 1 - \frac{v(t_r)}{c} = 1 - \frac{\alpha t_r / c}{\sqrt{1 + (\alpha t_r / c)^2}}, \quad (15)$$

which  $\rightarrow 0$  as  $t_r \rightarrow +\infty$ : the field ahead of an eternally accelerating mass is compressed without bound, the gravitational shadow of the Rindler horizon that the mass is forever falling away from. We pick this thread up in the companion note, where the horizon is made explicit; see also the Rindler geodesic figure in the notes directory.

## The Binary System

Linearized gravity is linear — that is the whole point — so the field of two masses is simply the sum of the fields of each. Let  $M_1$  and  $M_2$  trace worldlines  $\vec{x}_s^{(1)}(t)$  and  $\vec{x}_s^{(2)}(t)$ . The trace-reversed potential is the superposition of two Liénard–Wiechert terms of the form (10),

$$\bar{h}^{\mu\nu}(\vec{x}, t) = \frac{4G}{c^4} \sum_{A=1,2} M_A \frac{u_A^\mu u_A^\nu}{\kappa_A R_A} \frac{1}{c^2} \Big|_{t_r^{(A)}}, \quad (16)$$

each body contributing at its own retarded time  $t_r^{(A)} = t - |\vec{x} - \vec{x}_s^{(A)}(t_r^{(A)})|/c$ . Far from the system the two retardations and separations coalesce,  $R_A \rightarrow R$  and  $\hat{n}_A \rightarrow \hat{n}$ , and the slow-motion limit reduces the spatial potential to the second time derivative of the mass moment,

$$\bar{h}^{ij}(\vec{x}, t) = \frac{2G}{c^4 R} \ddot{I}^{ij}(t_r), \quad I^{ij}(t) = \sum_{A=1,2} M_A x_A^i(t) x_A^j(t), \quad (17)$$

so the interesting physics is carried not by the sum of the monopoles but by the time-dependence of the mass distribution.

Work in the centre-of-mass frame, where  $M_1 \vec{x}_s^{(1)} + M_2 \vec{x}_s^{(2)} = 0$ . This choice annihilates the mass dipole by construction, so — exactly as a single accelerating monopole could not radiate — the leading radiative multipole is the *quadrupole*. Writing  $M = M_1 + M_2$  and the reduced mass  $\mu = M_1 M_2 / M$ , the two positions are fixed by the relative coordinate  $\vec{r} = \vec{x}_s^{(1)} - \vec{x}_s^{(2)}$  alone,

$$\vec{x}_s^{(1)} = \frac{M_2}{M} \vec{r}, \quad \vec{x}_s^{(2)} = -\frac{M_1}{M} \vec{r}, \quad (18)$$

and the mass moment collapses onto the reduced one-body problem,

$$I^{ij} = M_1 \left( \frac{M_2}{M} \right)^2 r^i r^j + M_2 \left( \frac{M_1}{M} \right)^2 r^i r^j = \mu r^i r^j. \quad (19)$$

The radiated power is the quadrupole luminosity [2, 7]

$$L = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle, \quad Q_{ij} = \mu \left( r^i r^j - \frac{1}{3} r^2 \delta_{ij} \right), \quad (20)$$

the trace-free part of (19); the unequal masses enter only through  $\mu$ , so the radiation does not care how the mass is split, only about the reduced mass the splitting produces.

### Circular orbit

Place the orbit in the  $xy$ -plane with separation  $r$  and angular frequency  $\Omega$ , fixed by Kepler's third law  $\Omega^2 = GM/r^3$ . The relative coordinate and, through (18), the two worldlines are

$$\vec{r}(t) = r(\cos \Omega t, \sin \Omega t, 0), \quad (21)$$

$$\vec{x}_s^{(1)}(t) = \frac{M_2}{M} r(\cos \Omega t, \sin \Omega t, 0), \quad \vec{x}_s^{(2)}(t) = -\frac{M_1}{M} r(\cos \Omega t, \sin \Omega t, 0). \quad (22)$$

Feeding (21) into (19) and using the double-angle identities gives the moment tensor explicitly,

$$I^{ij}(t) = \mu r^2 \begin{pmatrix} \frac{1}{2}(1 + \cos 2\Omega t) & \frac{1}{2} \sin 2\Omega t & 0 \\ \frac{1}{2} \sin 2\Omega t & \frac{1}{2}(1 - \cos 2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

Only the oscillating part survives differentiation, and every time derivative brings down a factor  $2\Omega$ . The third derivative of the trace-free moment is therefore

$$\ddot{\ddot{Q}}_{ij} = 4\mu r^2 \Omega^3 \begin{pmatrix} -\sin 2\Omega t & \cos 2\Omega t & 0 \\ \cos 2\Omega t & \sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (24)$$

whose square is  $\ddot{\ddot{Q}}_{ij} \ddot{\ddot{Q}}_{ij} = 32 \mu^2 r^4 \Omega^6$  — a constant, so the time average in (20) is trivial. Assembling the pieces,

$$L = \frac{32}{5} \frac{G}{c^5} \mu^2 r^4 \Omega^6 = \frac{32}{5} \frac{G^4}{c^5} \frac{M_1^2 M_2^2 (M_1 + M_2)}{r^5}, \quad (25)$$

the second form following from Kepler's law. This is the energy loss that drives the inspiral, shrinks  $r$ , and chirps  $\Omega$  upward — the waveform LIGO hears. The luminosity is symmetric under  $M_1 \leftrightarrow M_2$  and peaks, for fixed total mass, at the equal-mass configuration  $M_1 = M_2$ ; a strongly unequal binary radiates feebly by comparison, its small companion barely stirring the field of the large one.

### General Keplerian orbit

Now let the orbit be an ellipse of semi-major axis  $a$  and eccentricity  $e$ , the relative coordinate tracing

$$r(\phi) = \frac{a(1 - e^2)}{1 + e \cos \phi}, \quad \vec{r} = r(\phi)(\cos \phi, \sin \phi, 0), \quad (26)$$

with  $\phi$  the true anomaly measured from periastron. The areal velocity is constant — Kepler's second law — so

$$r^2 \dot{\phi} = \sqrt{GM a(1 - e^2)} \equiv h, \quad (27)$$

which we use to trade time derivatives for  $\phi$  derivatives,  $\frac{d}{dt} = \frac{h}{r^2} \frac{d}{d\phi}$ . Carrying (26) through (19) and (20), the instantaneous power is

$$L(\phi) = \frac{8}{15} \frac{G^4}{c^5} \frac{M_1^2 M_2^2 (M_1 + M_2)}{a^5 (1 - e^2)^5} (1 + e \cos \phi)^4 \left[ 12(1 + e \cos \phi)^2 + e^2 \sin^2 \phi \right]. \quad (28)$$

The observable is the average over one orbital period  $T$ , and since  $dt = (r^2/h) d\phi$  the time average becomes a weighted integral over the true anomaly,

$$\langle L \rangle = \frac{1}{T} \int_0^T L dt = \frac{1}{T} \int_0^{2\pi} L(\phi) \frac{r^2}{h} d\phi. \quad (29)$$

The integrals are elementary — powers of  $(1 + e \cos \phi)$  against  $\cos \phi$  and  $\sin^2 \phi$ , each closing on a standard  $\int_0^{2\pi} \cos^n \phi d\phi$  — and collecting them yields the Peters–Mathews result [7],

$$\langle L \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{M_1^2 M_2^2 (M_1 + M_2)}{a^5} f(e), \quad f(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1 - e^2)^{7/2}}. \quad (30)$$

Setting  $e = 0$  recovers the circular law (25) with  $a = r$ , since  $f(0) = 1$ . The enhancement  $f(e)$  grows without bound as  $e \rightarrow 1$ : an eccentric binary dumps most of its energy in a brief, violent burst at periastron, where the bodies whip past at closest approach, and a highly eccentric orbit can outshine its circular counterpart of the same semi-major axis by orders of magnitude.

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