# Introduction to the Vicsek Model

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In this research primer you'll learn about how flocking can be described quantitatively, and be introduced to the tools you'll need to do independent research in the field. We begin with a brief historical account of the topic, and an overview of the types of questions that have opened up recently for researchers. Some of these are being answered as we speak, others are ripe for the taking. In the theory and problems section we outline the basic mathematics needed, and propose different activities you should do to familiarize yourself with the tools needed to make progress on the above questions. As you work on these, you will undoubtedly discover new questions that you cannot answer about math, physics, information theory, computer science, biology, ecology, ... you get the gist. Write these down! Keep a small journal full of questions, and then talk about them. Usually your questions will already have answers — you just need to be pointed in the right direction — other times you will see that your questions are doors that will lead you to an undiscovered place that you will have to tame if you are to bring others there behind you. Now, let's start on our journey!

## History and Overview

Describing flocking behavior in the context of computer animation started in 1987 by Craig Reynolds[1]. He described the minimal model of a flocking element, a simulated *bird-oid*, coining the term *boid*. Nearly a decade later, the Hungarian physicist Tamás Vicsek and his colleagues analyzed numerically the thermodynamic limit of boid flocks. In their 1995 paper they showed that flocks exhibit a phase transition controlled by a noise parameter [2], analogous to the emergence of spontaneous magnetization in a ferromagnet as the temperature is lowered. Above a critical noise value boids are individualistic, moving this way and that; below the critical value they begin to correlate their behavior and a collective dynamics emerges [Video]. For the honor of this insight, the model now bears his surname.

The analogy to magnetism is apt, since the Vicsek model is an extension of the well known XY-model; the velocity vectors of the boids are like the spins there, except rather than being confined to a lattice they are allowed to move around. The XY model contains the famous Berezinskii–Kosterlitz–Thouless phase transition in which vortex-antivortex binding describes a universality class that includes multiple condensed matter systems. The exact nature of the flocking phase transition is a mystery; namely, what are the critical exponents that define the universality class of the model? Vicsek and his colleage András Cziróc provided evidence for a second-order phase transition, though others would dispute this for a time[3, 4]. In 1998, American physicists John Toner and Yuhai Tu discovered the continuum limit of the Vicsek Model [5, 6]. Using field theoretic machinery, they constructed the renormalization group flow for their coupling constants and extracted the critical exponents in the case of 2 dimensions; for this amazing work the continuum model now bears their names.

By 2007 it was known that the noise transition is second-order; then Vicsek's student Máté Nagy revealed more richness in the model— a second phase transition, controlled by the velocity. This one is of first-order and heralded by the appearance of global density waves[7], leading to the current idea that the VM describes a kind of liquid-gas phase transition[8]. A good review of the physics found in the model is by Ginelli [9]. Others began to look at generalizations of the model: adding acceleration[10] and changing the nature of the connections [11], addressing the question

of whether the model actually describes real flocking[12, 13]. Using data ingeniously gathered by aiming multiple cameras on flocks of starlings, and a ton of registration processing, William Bialek et al found that the connections between neighbors do not have a distance cut-off and are therefore topological.

Interpreted as a capacity to *pay attention* to other flockers, the topological VM refocused analysis on the individual as a data-collecting and information-processing agent, starting up the information-theoretic narrative of flocking[14, 15, 16, 17]. How does information flow through flocks? How much does the motion of an individual predict the motion of the flock, and vice versa? How large do clusters of flockers have to be to shepherd the whole flock? These go hand in hand with graph-theoretic questions concerning the structure and temporal evolution of the adjacency matrix describing a flock[18]. How do the degree statistics evolve? What is the relationship between clustering, and other metrics, on both sides of the phase transition? There are even computation-theoretic questions on the nature of information processing by the flock[19]. Is there a sense in which a flock performs a computation? What is that computation, and how do we access the resulting output? There's still a lot of unexplored territory.

## Theory and Problems

Whenever starting a new research journey it's smart to first reproduce known results— the process helps orient our thinking — so let's do that. Read Vicsek's original paper, and spend extra time processing the plots in your mind. We're going to reproduce them which means you're going to have to code up the Vicsek model; this primer will guide you through that.

Use whatever coding language you want, with the foreknowledge that what you write will have to be scalable. The thermodynamic limit requires simulating ever larger and larger flocks, so use a language with vectorized operations. Always be thinking about how you can reduce the computational complexity of implementing the dynamics. Get comfortable with writing visualization subroutines; when simulating systems in real time you can spot bugs really quickly by simply observing some $\mathcal{O}(10-100)$ timesteps. Code with modularity in mind; DO NOT write one long piece of code that does all the steps you want. Write routines, write subroutines that are called by these routines, construct objects that talk to each other. That way it's much easier to reuse code you've already written in the future. Lastly, and I cannot stress this enough, document your code; there should be a sentence explaining what the code is doing every few lines. Your future self will thank you for this.

### Theory

The Vicsek model consists of $N$ boids in a domain $[0, L]^d$ with cyclic boundary conditions. Each boid is described by position and velocity degrees of freedom, $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$, respectively, with $i = 1, 2, \ldots, N$, and the speed of each boid constant, $||\boldsymbol{v}_i|| = v$. The position DoF obeys the equation of motion

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i, \tag{1}$$

but the velocity EoM cannot be written as a differential equation. In words, each boid looks

at the velocity of all the boids within a radius $r$ of it, and tries to compute the mean. This mean computation is where the stochasticity enters an otherwise deterministic model, as the boid's computed mean orientation falls uniformly within a (solid) angle of radius $\pi\eta$ of the actual mean. Here $\eta \in [0, 1]$ is the noise parameter (note that in Vicsek's paper the noise parameter includes the factor of $\pi$).

We implement a temporal discretization by choosing a timescale $\Delta t$, and use a leapfrog scheme where the positions are defined on temporal slices with $t_n = n\Delta t$, while the velocities are defined on temporal slices in between those with $t_{n+\frac{1}{2}}$. Then the discretized position evolution is

$$\boldsymbol{r}_i(t + \Delta t) = \boldsymbol{r}_i(t) + \Delta t\ \boldsymbol{v}_i(t + \frac{\Delta t}{2}), \tag{2}$$

which introduces errors at $\mathcal{O}(\Delta t^3)$ (Prove this!). The velocity is updated by first normalizing the sum of all the velocities within a radius $r$ of the boid,

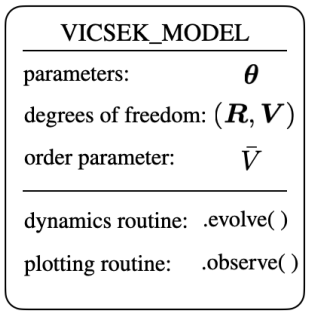
$$\hat{\boldsymbol{n}}_i \propto \sum_{j=1}^{N} \Theta(r - ||\boldsymbol{r}_j(t) - \boldsymbol{r}_i(t)||)\boldsymbol{v}_j(t - \frac{\Delta t}{2}), \tag{3}$$

where $\Theta$ is the Heaviside function acting as a kernel. Then we draw a random vector from a uniform distribution on the subset of the unit sphere with mean $\hat{\boldsymbol{n}}_i$ and dispersion $\pi\eta$, and update the velocity as

$$\boldsymbol{v}_{i+\frac{\Delta t}{2}} = v\hat{\boldsymbol{N}} \qquad \text{where} \qquad \hat{\boldsymbol{N}} \sim \mathcal{U}_{S^d}(\hat{\boldsymbol{n}}_i, \pi\eta). \tag{4}$$

That's it, now we just have to implement all this and start running simulations. I've been general here keeping the number of dimensions arbitrary, so you're going to have to figure out how to translate all this math for $d = 2$ while keeping in mind all the computational constraints we spoke of earlier.

To start note that the model, as stated above, is defined by the set of parameters $\boldsymbol{\theta} = \{N, L, v, r, \eta, \Delta t\}$. For reference, we will structure our code as an object that runs the model:




This diagram is not definitive, and you may want to add other parameters or routines to the class. Code up a class, VISCEK__MODEL, that instantiates the model when passed the set of parameters $\boldsymbol{\theta}$. Instantiation must construct the degrees of freedom. I recommend two $N \times 2$ arrays, one containing all the positions of the boids, $\boldsymbol{R}$, and the other their velocities, $\boldsymbol{V}$. The first can be

drawn uniformly from the domain $[0, L]^2$, while the second can be drawn uniformly from the circle of radius $v$. Once the velocities instantiate, compute the order parameter, $\bar{V}$. You may want to construct a subroutine that does this within the class, since you'll be computing the order parameter many times.

Next we need the dynamics. Add a routine to your class called EVOLVE(). When called, it should perform the update rules described earlier. Pay attention to any calculations that you perform repeatedly, constructing subroutines for them. Since you're going to want this code to scale, make sure it is vectorized: There should be no nested FOR loops throughout. This should take a while, so don't fret if you fin yourself getting stuck. There are nuances that you will discover that are annoying to deal with.

Lastly, add a routine called called OBSERVE(). The first time this routine is called it should construct an image of the simulation i.e. it should show the domain, and plot the positions of the boids. You can be fancy and include the directions the boids are facing as well. Further calls of the routine should not create a new image, but update the data contained in the image that already exists. If that image was closed, then calling the routine should be treated as the first time.

Come back later for new parts that have been added to this primer. Good luck!

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