Graph Theory

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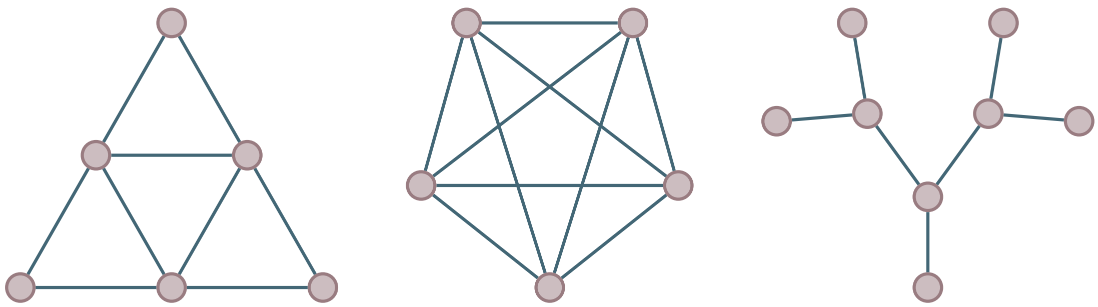
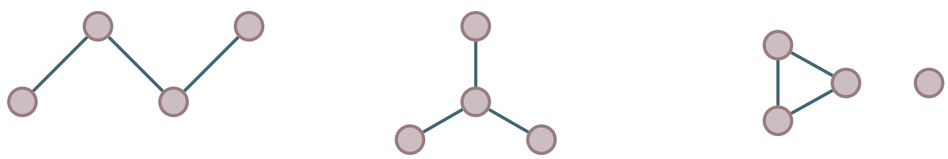
Outline

Since a picture is worth a thousand words, lets look at a few: These structures are **graphs**. They

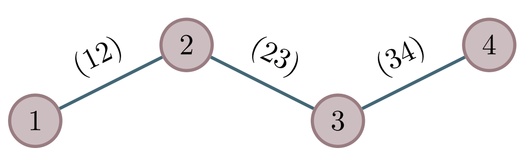


consist of **vertices** (or *nodes*) and **edges** connecting vertices. We call the collection of all vertices the **vertex set**, V, and the collection of all the edges the **edge set**, E. The number of vertices is typically denoted $N = |V|$, while the number of edges is usually left as simply $|E|$. A particular node or edge is referenced with lowercase i.e. $v \in V$ and $e \in E$.

Knowing that a graph has $|V|$ vertices and $|E|$ edges is not enough information to reconstruct it. Each of these graphs has four vertices and three edges. To reconstruct a graph you need to be



told which edges attach to which vertices. There are two standard ways to do this, and you will encounter either often enough to familiarize yourself with both. The first is to label the vertices with numbers, and then label the edges based on your labeling scheme for the vertices. The label for the edge connecting vertex v with v' is the tuple (v, v'). Using this scheme the first graph above would be labeled Now the sets $V = \{1, 2, 3, 4\}$ and $E = \{(12), (23), (34)\}$ have enough information



to determine the graph, and often you'll see this written as $G = \langle V, E\rangle$. Notice that if someone labels the vertices differently than you, their E might look different but continue to describe the same graph.

For now we do not allow edges to connect to the same vertex, or multiple edges between the same two vertices. The second way to describe which edges are attached to which vertices is through an **incidence matrix**, $\boldsymbol{N}$. The matrix is a $|V| \times |E|$ array, each row representing one of the vertices and each column representing one of the edges. If vertex v is one of the endpoints of edge e then we fill in the array with a 1, otherwise it gets a 0. The next page has some examples of incident matrices for some of the graphs above (the 0's have been omitted for clarity). Notice that even

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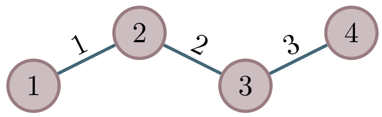
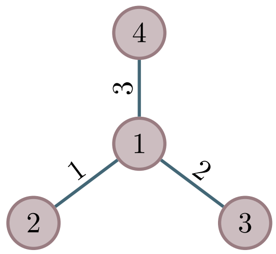


vertices \ edges	1	2	3
1	1		
2	1	1	
3		1	1
4			1



vertices \ edges	1	2	3
1	1	1	1
2	1		
3		1	
4			1

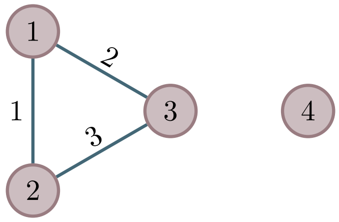
though all three graphs have the same number of edges and vertices, their incidence matrices differ. See if you can write the incidence matrix for a more complicated graph.

If you stare at incidence matrices long enough, you should notice some patterns. Since every edge is always connected to two vertices (its endpoints), every column sum must equal 2. Every row sum, on the other hand can differ, and is equal to the number of edges attached to each vertex. We call this number the **degree** of the vertex, and denote it as $\deg(v)$. If we add up all the column sums, we get $2|E|$. This is equal to adding up all the row sums, and we arrive at

$$\sum_{v \in V} \deg(v) = 2|E|. \tag{1}$$

When we count all the edges coming out of all the vertices we end up counting each edge twice.

Since we have a matrix at our disposal, can we use it to make other matrices? Unless the number of edges and vertices are equal, the incidence matrix will not, in general, be square. So, to multiply two copies of $\boldsymbol{N}$, we need to take the transpose of one of them (written in the usual way, $\boldsymbol{N}^T$). Depending on the order in which we multiply, we will get either a $|V| \times |V|$ matrix or an $|E| \times |E|$ matrix. Let's examine the former first. Expanding the matrix multiplication in terms of



vertices \ edges	1	2	3
1	1	1	
2	1		1
3		1	1
4			

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components,

$$\begin{aligned}[\boldsymbol{N}\boldsymbol{N}^T]_{vv'} &= \sum_{e\in E}[\boldsymbol{N}]_{ve}[\boldsymbol{N}^T]_{ev'} \\ &= \sum_{e\in E}[\boldsymbol{N}]_{ve}[\boldsymbol{N}]_{v'e}.\end{aligned}$$

First consider the diagonal components, when $v' = v$. Then the summand is 1 for each edge connected to v and 0 otherwise; the sum is simply $\deg(v)$. When $v' \neq v'$ then both terms in the summand are non-zero if and only if $e = (vv')$. The information about which vertices are connected by a single edge is organized into the **adjaceny matrix**, $\boldsymbol{A}$, for which $[\boldsymbol{A}]_{vv'} = 1$ when v and v' are connected by an edge and 0 otherwise. The most useful way to organize the degrees of a graph is in a diagonal $|V| \times |V|$ array called the **degree matrix**, $\boldsymbol{D}$, for which $[\boldsymbol{D}]_{vv} = \deg(v)$. With these we have

$$\boldsymbol{N}\boldsymbol{N}^T = \boldsymbol{A} + \boldsymbol{D} \tag{2}$$

Verify that this holds for the examples above.

More generally, if you can hop from one vertex to another along k connected edges, then the path has length k. If there is a path of length $k > 0$ between any two vertices in V, then the graph is **path-connected**. Convince yourself that $\boldsymbol{A}^k$ counts the number of paths of length k.