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In 1922 Alexander Friedmann derived, from Einstein's field equations, that a homogeneous and isotropic universe cannot sit still — it must expand or contract, at a rate dictated entirely by what fills it [1]. What fills it enters through a single number, the equation-of-state parameter w, and the aim of these notes is to compute that number from first principles for two very different gases and then watch how each drives the expansion.

We proceed in four movements. First we compute the stress-energy of a homogeneous, isotropic photon gas and find $w = 1/3$. Next we treat a gas of massive particles and recover $P = \frac{1}{3}\rho v^2$ — a result that contains both the photon gas, as $v \to c$, and pressureless dust, as $v \to 0$. We then turn to the cosmological constant, whose equation of state $w = -1$ follows from symmetry alone rather than from any microscopic average. With these three equations of state in hand, we feed the barotropic relation $P = w\rho c^2$ into the Friedmann equations and solve for the scale factor, specializing in turn to radiation, matter, and the cosmological constant.

Throughout we keep SI units and carry every factor of c explicitly. The metric signature is $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, and ρ denotes the mass density, so that the perfect-fluid stress-energy reads $T^{\mu\nu} = \mathrm{diag}(\rho c^2, P, P, P)$ and the equation-of-state parameter is $w = P/(\rho c^2)$.

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Photon Gas

The electromagnetic stress-energy tensor

Light carries energy and momentum, so a box of it gravitates — and to know how, we need its stress-energy tensor. For the electromagnetic field it reads [2]

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{1}$$

where the SI normalization is chosen so that every component carries units of energy density. Everything flows from the field-strength tensor itself, which in the signature $(+,-,-,-)$ reads

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{2}$$

the factors of $1/c$ scaling the electric entries so that $F^{\mu\nu}$ is dimensionally uniform. Carrying out the contractions in (2), the tensor splits into three familiar pieces — the electromagnetic energy density (3), the Poynting flux (4), and the Maxwell stress tensor (5):

$$T^{00} = \tfrac{1}{2}\left(\epsilon_0|\boldsymbol{E}|^2 + \tfrac{1}{\mu_0}|\boldsymbol{B}|^2\right), \tag{3}$$

$$T^{0i} = \frac{1}{\mu_0 c}\,\epsilon_{ijk}E_jB_k, \tag{4}$$

$$T^{ij} = \tfrac{1}{2}\left(\epsilon_0|\boldsymbol{E}|^2 + \tfrac{1}{\mu_0}|\boldsymbol{B}|^2\right)\delta_{ij} - \epsilon_0 E_iE_j - \frac{1}{\mu_0}B_iB_j. \tag{5}$$

The factor $1/c$ in $T^{0i} = S_i/c$ (with $S_i = \frac{1}{\mu_0}\epsilon_{ijk}E_jB_k$ the Poynting vector) ensures that T^{0i} shares the energy-density units of the rest of the tensor.

Isotropic averaging

A single field configuration picks out a direction, but a thermal sea of photons does not — so to describe the gas we must average over orientations. Decompose the electric field (and likewise the magnetic field) into spherical components,

$$\begin{aligned} E_x(B_x) &= E(B)\,\sin\theta\cos\phi, \\ E_y(B_y) &= E(B)\,\sin\theta\sin\phi, \\ E_z(B_z) &= E(B)\,\cos\theta. \end{aligned}$$

Since we are dealing with an isotropic photon gas we average the stress tensor over all directions by integrating over the solid angle,

$$\langle T^{\mu\nu}\rangle = \frac{1}{4\pi}\int d\Omega\; T^{\mu\nu}. \tag{6}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$. The diagonal averages all evaluate to $\frac{1}{3}$; for example,

$$\langle \sin^2\theta\cos^2\phi\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} \sin^2\theta\cos^2\phi\ \sin\theta\, d\phi\, d\theta = \frac{1}{4\pi}\underbrace{\left(\int_0^{2\pi}\cos^2\phi\, d\phi\right)}_{=\pi}\underbrace{\left(\int_0^{\pi}\sin^3\theta\, d\theta\right)}_{=4/3} = \frac{1}{3}, \tag{7}$$

and identically $\langle \sin^2\theta\sin^2\phi\rangle = \langle\cos^2\theta\rangle = \frac{1}{3}$. The off-diagonal averages vanish, since each contains an odd power of a sinusoid over a full period, e.g.

$$\langle \sin^2\theta\sin\phi\cos\phi\rangle = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin^2\theta\ \sin\phi\cos\phi\ \sin\theta\, d\phi\, d\theta = 0, \tag{8}$$

because $\int_0^{2\pi}\sin\phi\cos\phi\, d\phi = 0$. With these,

$$\langle T^{00}\rangle = \tfrac{1}{2}\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right), \tag{9}$$

$$\langle T^{0i}\rangle = 0, \tag{10}$$

$$\langle T^{ij}\rangle = \tfrac{1}{6}\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right)\delta_{ij}. \tag{11}$$

Energy density, pressure, and equation of state

The averaged tensor is diagonal, so we may equate it to the stress-energy of a perfect fluid, $T^{\mu\nu} = \mathrm{diag}(\rho c^2, P, P, P)$. Writing the field energy density as $u = \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0)$, we identify $\rho_\gamma c^2 = u$ and read off the mass density and pressure of an isotropic, homogeneous photon gas:

$$\rho_\gamma = \frac{1}{2c^2}\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right), \tag{12}$$

$$P_\gamma = \tfrac{1}{6}\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right). \tag{13}$$

From these we read off the equation of state for the photon gas,

$$w = \frac{P}{\rho_\gamma c^2} = \frac{1}{3}. \tag{14}$$

Voila — the radiation value, obtained without ever invoking a temperature or a Planck spectrum. It follows purely from the tracelessness of the electromagnetic stress-energy tensor.

Ideal Gas

Stress-energy of a collection of particles

Photons are massless and always travel at c; now we ask what changes when the constituents carry mass and may move at any speed. Consider the stress-energy tensor of a collection of N particles with 4-momenta p_n^μ and mass m:

$$T^{\mu\nu} = \sum_{n=0}^{N}\int d\tau\ p_n^\mu u_n^\nu\, \frac{\delta^4(x^\alpha - x_n^\alpha(\tau))}{\sqrt{|g|}} = m\sum_{n=0}^{N}\int \gamma_n\, dt\ v_n^\mu v_n^\nu\, \frac{\delta(t-t_n)\,\delta^3(\boldsymbol{x} - \boldsymbol{x}_n(t))}{\sqrt{|g|}}. \tag{15}$$

We can read the meaning of this expression by contracting with the 4-velocity u_ν of a stationary observer to measure the total momentum of the gas as that observer sees it. With $u_\mu u^\mu = 1$,

$$p^\mu_{\text{gas}} = u_\nu T^{\mu\nu} = m \sum_{n=0}^{N} \gamma_n v^\mu_n(t_n, \boldsymbol{x}) \, \frac{\delta^3(\boldsymbol{x} - \boldsymbol{x}_n(t))}{\sqrt{|g|}}, \tag{16}$$

which is simply the sum of the 4-momenta of all the particles — exactly what a stress-energy tensor ought to deliver.

Continuum limit

Taking $N \to \infty$ lets us trade the discrete sum for a smooth field. We define the (relativistic) mass density as the local sum of moving masses per proper spatial volume,

$$\rho(\boldsymbol{x}, t) \;=\; \lim_{N\to\infty} \sum_{n=0}^{N} m\, \gamma_n \, \frac{\delta^3(\boldsymbol{x} - \boldsymbol{x}_n(t))}{\sqrt{|g|}}, \tag{17}$$

where the factor γ_n counts each particle's relativistic mass $m\gamma_n$ rather than its rest mass — this is precisely the weighting that makes $T^{00} = \rho c^2$ the energy density. With this definition,

$$\begin{aligned} T^{\mu\nu} &\to \lim_{N\to\infty} \int dt \sum_{n=0}^{N} m \, \frac{\delta^3(\boldsymbol{x} - \boldsymbol{x}_n(t))}{\sqrt{|g|}} \, \gamma_n \, v^\mu_n v^\nu_n \, \delta(t - t_n) \\ &= \int dt \int d^3\boldsymbol{x}' \, \rho(\boldsymbol{x}', t') \, v^\mu(\boldsymbol{x}', t') v^\nu(\boldsymbol{x}', t') \, \delta^3(\boldsymbol{x} - \boldsymbol{x}') \, \delta(t - t') \\ &= \rho(\boldsymbol{x}, t) \, v^\mu(\boldsymbol{x}, t) \, v^\nu(\boldsymbol{x}, t). \end{aligned} \tag{18}$$

Isotropy and the equation of state

Assuming the gas is homogeneous and isotropic, the density is uniform and the velocities may be averaged over direction. With the 4-velocity $v^\mu = \gamma(c, \boldsymbol{v})$ and $v_i = v\,(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$, the angular average gives

$$\langle T^{\mu\nu} \rangle = \frac{1}{4\pi} \int d\Omega \; T^{\mu\nu} = \rho \, \text{diag}\Big(c^2, \tfrac{v^2}{3}, \tfrac{v^2}{3}, \tfrac{v^2}{3}\Big), \tag{19}$$

so that $\langle T^{00} \rangle = \rho c^2$ is the relativistic energy density. The gas is therefore a perfect fluid with

$$P = \tfrac{1}{3}\rho v^2, \qquad w = \frac{P}{\rho c^2} = \frac{v^2}{3c^2}. \tag{20}$$

Two limits are worth dwelling on. As $v \to c$ we recover $w = \frac{1}{3}$ — the very value the Photon Gas section extracted from the field tensor, now reached from the opposite direction, by speeding particles up until they are effectively photons. In the classical limit $v \to 0$ the pressure vanishes and the gas becomes pressureless dust. One expression, $w = v^2/3c^2$, interpolates the entire range from cold matter to radiation.

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Dark Energy

The vacuum equation of state

The two gases so far were made of *stuff* — fields or particles moving through spacetime. The cosmological constant is different in kind: it is the energy of empty space itself, and its equation of state follows not from a microscopic average but from a symmetry.

Demand that the vacuum look identical to every inertial observer. Its stress-energy tensor must then be invariant under all Lorentz boosts, and the only rank-two tensor with that property is the metric itself. Hence

$$T^{\mu\nu}_{\rm vac} = -\,\rho_{\rm vac}\, c^2\, g^{\mu\nu}, \tag{21}$$

with a single constant $\rho_{\rm vac}$ setting the scale. This is precisely the object Einstein introduced by hand: moving the cosmological term to the right-hand side of the field equations, $G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} - \Lambda g^{\mu\nu}$, identifies it as

$$T^{\mu\nu}_{\rm vac} = -\frac{\Lambda c^4}{8\pi G}\, g^{\mu\nu}, \qquad \Longrightarrow \qquad \rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G}. \tag{22}$$

Reading (21) against the perfect-fluid form $T^{\mu\nu} = \mathrm{diag}(\rho c^2, P, P, P)$ in flat space, where $g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, gives

$$\rho c^2 = \rho_{\rm vac} c^2, \qquad P = -\,\rho_{\rm vac} c^2, \tag{23}$$

so the vacuum pushes outward with a pressure equal and opposite to its energy density. Its equation of state is therefore

$$w = \frac{P}{\rho c^2} = -1. \tag{24}$$

Indeed, this is the one value of w that no gas of moving particles can reach — $w = v^2/3c^2$ is bounded below by zero — which is exactly why a cosmological constant cannot be mimicked by ordinary matter or radiation.

Cosmic Expansion

Friedmann equations for a barotropic fluid

With an equation of state in hand, we can finally ask Friedmann's question: how does the universe respond? Consider an FRW spacetime filled with a fluid obeying the barotropic equation of state

$$P = w\rho c^2. \tag{25}$$

The Friedmann equations for the (spatially flat) scale factor are [3]

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\,\rho, \tag{26}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) = -\frac{4\pi G}{3}\,\rho\,(1+3w). \tag{27}$$

Combining these eliminates ρ and yields a single equation for a,

$$a\ddot{a} = -k\,\dot{a}^2, \qquad k = \frac{1+3w}{2}. \tag{28}$$

Power-law and exponential solutions

Since each side carries the same number of derivatives, for $w \neq -1$ we try a power-law ansatz $a(t) \propto t^n$:

$$n(n-1) = -kn^2 \quad \Longrightarrow \quad n = \frac{1}{1+k} = \frac{2}{3(1+w)}, \tag{29}$$

so that

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{30}$$

For $w = -1$ the relation $a\ddot{a} = \dot{a}^2$ gives $\frac{d}{dt}(\dot{a}/a) = 0$, whence

$$a(t) \propto e^{H_0 t}. \tag{31}$$

Here $H_0 = \dot{a}/a$ is constant, since for $w = -1$ the density ρ_{vac} does not change. The first Friedmann equation fixes its value,

$$H_0 = \sqrt{\frac{8\pi G}{3}\,\rho_{\text{vac}}} = \sqrt{\frac{\Lambda c^2}{3}}, \tag{32}$$

the constant Hubble rate toward which a Λ-dominated universe asymptotes.

Specializing to each gas

Each gas now slots into this single solution. Radiation — the photon gas, or the ideal gas driven to $v \to c$ — has $w = \frac{1}{3}$, so that $a(t) \propto t^{1/2}$. Matter, the same gas in its cold limit $v \to 0$, has $w = 0$ and expands as $a(t) \propto t^{2/3}$. A cosmological constant sits at the boundary of the power-law family, $w = -1$, where the algebra hands us $a(t) \propto e^{H_0 t}$ instead — the de Sitter expansion that runs away exponentially at the constant Hubble rate H_0. Three gases, three histories, all read off from the lone exponent $2/3(1+w)$ and its breakdown at $w = -1$.

Connecting to the observed expansion history

These three solutions are not merely a menu of possibilities — our own universe has lived through all three in succession, and the transitions are written into the light we receive. To read them we need the bridge between the scale factor and the one quantity astronomers actually measure: redshift.

A photon emitted when the scale factor was a and received today, when it is a_0, is stretched along with space itself. Normalizing $a_0 = 1$ today, the relation is

$$1 + z = \frac{a_0}{a} = \frac{1}{a}, \tag{33}$$

so that high redshift means a small, young universe and $z = 0$ is here and now.

How each component fares as the universe expands follows from conservation of stress-energy, $\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P/c^2)$, which for $P = w\rho c^2$ integrates to $\rho \propto a^{-3(1+w)}$. Substituting (33) and the three values of w,

$$\rho_r \propto a^{-4} = (1+z)^4 \qquad \text{(radiation, } w = \tfrac{1}{3}\text{)}, \tag{34}$$

$$\rho_m \propto a^{-3} = (1+z)^3 \qquad \text{(matter, } w = 0\text{)}, \tag{35}$$

$$\rho_\Lambda = \text{const} \qquad \text{(dark energy, } w = -1\text{)}. \tag{36}$$

Radiation dilutes one power of $(1+z)$ faster than matter — three powers from the expanding volume, plus one more because each photon's own energy redshifts away. Dark energy, being the energy of space itself, does not dilute at all.

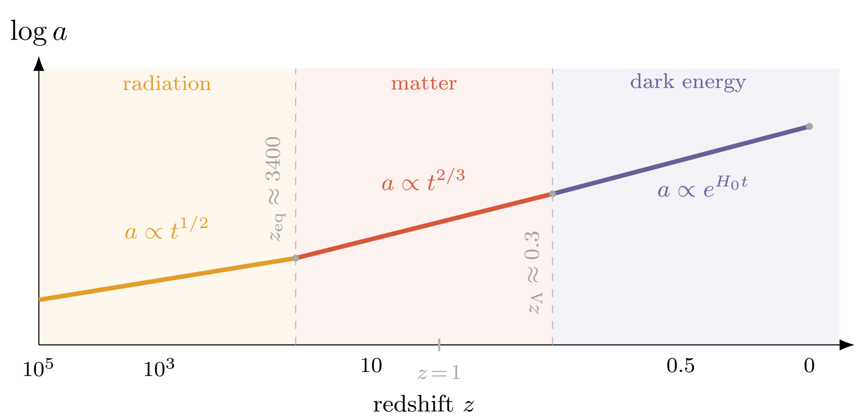
Because the three scale so differently, whichever is densest changes as the universe ages, and the cosmos passes through three epochs in turn. At early times radiation dominates and $a(t) \propto t^{1/2}$. The Planck 2018 parameters [4] place matter–radiation equality, $\rho_r = \rho_m$, at

$$1 + z_{\text{eq}} = \frac{\Omega_m}{\Omega_r} \approx 3400, \tag{37}$$

after which matter takes over and $a(t) \propto t^{2/3}$. Far later, with present density fractions $\Omega_m \approx 0.315$ and $\Omega_\Lambda \approx 0.685$, the constant dark-energy term overtakes the thinning matter at

$$1 + z_\Lambda = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} \approx 1.3, \qquad z_\Lambda \approx 0.3, \tag{38}$$

and the expansion crosses over toward the exponential $a(t) \propto e^{H_0 t}$ of de Sitter space. (The acceleration itself, $\ddot{a} > 0$, sets in a little earlier, near $z \approx 0.6$, once $w_{\text{tot}} < -\frac{1}{3}$.) The same three numbers $w = \frac{1}{3}$, 0, -1 that we computed from a field, a gas, and a symmetry thus order the entire history of cosmic expansion — radiation first, then matter, and finally the dark energy that governs it today.



The figure above sketches this history of the scale factor against redshift, with the early universe (high z) on the left and today ($z = 0$) on the right. The horizontal axis is split at $z = 1$: logarithmic in $\log(1+z)$ for $z > 1$, compressing the long early history out to $z \sim 10^5$, and linear in z for $z < 1$, expanding the recent, dark-energy–dominated epoch. The three eras are drawn with comparable widths. The curve is a single continuous track — each era's segment begins where the previous one ends, at matter–radiation equality ($z_{\mathrm{eq}} \approx 3400$) and matter–$\Lambda$ equality ($z_\Lambda \approx 0.3$) — with the dominant expansion law shifting from $a \propto t^{1/2}$ (radiation, amber) to $a \propto t^{2/3}$ (matter, red) to $a \propto e^{H_0 t}$ (dark energy, purple). The spacing is schematic; the redshifts are from the Planck 2018 parameters [4].

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